



2.3.4 Linear combinations of vectors

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Introduction

- In this topic, we will
 - Define linear combinations of vectors
 - Look at questions that may be asked about such combinations
 - Describe some special cases as averages and weighted averages





Linear combinations

- Given a vector space over a field $\mathbf{F}$,
given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in that vector space, and
 m scalars $\alpha_1, \dots, \alpha_m$ all in $\mathbf{F}$
we will call

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m$$

a *linear combination* of these m vectors and corresponding scalars

- We may also write this as

$$\sum_{k=1}^m \alpha_k \mathbf{v}_k$$





Vector spaces

- For example, this is a linear combination of five vectors in $\mathbf{R}^4$

$$2.3 \begin{pmatrix} 8 \\ 5 \\ 6.5 \\ 2.3 \end{pmatrix} - 4.7 \begin{pmatrix} 4.6 \\ 5.7 \\ 0.3 \\ 2.9 \end{pmatrix} + 5.9 \begin{pmatrix} 2.2 \\ 9.9 \\ 7.3 \\ 4.4 \end{pmatrix} - 8.6 \begin{pmatrix} 0.2 \\ 4.8 \\ 8.8 \\ 9.8 \end{pmatrix} + 0.1 \begin{pmatrix} 3.8 \\ 1.7 \\ 4.6 \\ 1.9 \end{pmatrix} = \begin{pmatrix} 8.42 \\ 2.01 \\ -18.61 \\ -66.47 \end{pmatrix}$$





Linear combinations are finite

- A linear combination must be made up of finitely many vectors: we cannot take a linear combination of infinitely many vectors, as the sum may simply not exist
- In general, none of the vectors in a linear combination will be the zero vector, for if

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_m \mathbf{v}_m = \mathbf{u}$$

then

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_m \mathbf{v}_m + \beta \mathbf{0} = \mathbf{u}$$

for all possible scalar values of β





Finding linear combinations

- One of the most significant questions we will ask in this course is given a target vector $\mathbf{u}$ in a particular vector space, and given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in that vector space,
do there exist m scalars $\alpha_1, \dots, \alpha_m$ all in $\mathbf{F}$ such that
$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m ?$$
- Using the terminology of linear combinations,
given a target vector $\mathbf{u}$ and m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in the same vector space, is there a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ that equals the target vector?





All possible linear combinations

- Another question we may ask is
given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in that vector space,
what are some of the properties of all possible linear
combinations of these vectors?





A single vector

- If $m = 1$, then we are considering
linear combinations of a single vector $\mathbf{v}_1$; that is,
we are simply considering possible scalar multiples of $\mathbf{v}_1$
- In this case,
 $\mathbf{u} = \alpha_1 \mathbf{v}_1$ if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}_1$
All possible scalar multiples $\alpha_1 \mathbf{v}_1$ forms a line in the vector space





Averages

- Now, given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in a vector space,

if $\alpha_1 = \dots = \alpha_m = \frac{1}{m}$, then

$$\frac{1}{m} \mathbf{v}_1 + \dots + \frac{1}{m} \mathbf{v}_m$$

is the average of the m vectors





Weighted averages

- Now, given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in a vector space,

if $\alpha_1 + \dots + \alpha_m = 1$, then

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m$$

is a *weighted* average of the m vectors





Summary

- Following this topic, you now
 - Have a deeper understanding of linear combinations
 - Are aware of some questions we will ask in this course
 - Understand the ideas of averages and weighted averages of vectors





References

- [1] https://en.wikipedia.org/wiki/Linear_combination





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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